A Variation of Discrete Silverman's Game with Varying Payoffs

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A discrete form of Silverman's game, a two-player zero sum game, is played with each player choosing a number from 1 to n. Each player's goal is to choose the larger number as long as it is less than three times the opponent's chosen number. Here we consider a variation of Silverman's game, wherein the payoff to the player choosing the larger number is the difference between the two numbers, but if the larger number is at least three times the smaller number, the payoff to the player choosing the smaller number is twice the difference between the two numbers. Analysis of payoff matrices and the Minimax Theorem are used to solve the game. In both versions of the game, results show that when $n \geq 3$ the unique optimal strategy reduces each player's choices to just three numbers. The difference between the two solutions is in the probabilities each player must select the three choices.

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INTRODUCTION

Silverman's game was first introduced by RJ Evans (1979). Silverman's game is a variation of Mendelson's game. In its general form, Silverman's game is a two player zero-sum game wherein each player selects one number from a given set of numbers, which may be continuous (i.e. from an interval) or discrete. The player who selected the larger number wins 1 unit, unless the larger number is at least c times as large, in which case the player who selected the smaller number wins an amount equal to b units. If the numbers selected are equal, the payoff is 0. Studies have been conducted on the continuous and discrete cases, for example Evans and Heuer (1992), Heuer and Leopold-Wildburger (1993), Heuer (2001).

A basic form of Silverman's games has both players choosing from identical discrete sets consisting of the integers from 1 to n. In this version, $c = 3$ and $b = 2$. In other words, the player who selected the larger number wins 1 provided it is less than 3 times as large as the other number, otherwise the player who selected the smaller number wins 2. This version was presented in the Game Theory lecture notes of Ferguson (2000).

In this paper, we consider the following variation of

Silverman's game: the multiplier is set at $c = 3$ as with the basic form of the game, so that the player who selected the larger number wins provided it is less than 3 times as much as the smaller number, otherwise the player who selected the smaller number wins. The payoff, however, is the difference between the two numbers if the larger number wins, or twice the difference (in absolute value) if the smaller number wins.

Basic Silverman's Game

Playing the basic form of Silverman's game, two players each choose one number simultaneously from the integer set {1, 2, …, n}. The resulting payoff matrix of possible gains and losses is skew-symmetric, implying that the game is symmetric. A symmetric game is fair and therefore its value $v = 0$, because one player does not have an advantage over the other and they share the same set of strategic choices (Ferguson, 2000). It also follows that both players have the same optimal strategies.

The resulting payoff matrix is typically solved using linear programming. Large payoff matrices may be simplified using dominated strategies. A strategy is dominated by another strategy if it is never profitable to select that strategy over the dominating strategy. In terms of the payoff matrix, a column is dominated by another column if each entry in the dominated column is greater than or equal to the corresponding entry in the dominating column, i.e.

$$a_{ij} \geq a_{ik} \text{ for all } i \qquad (1)$$

hence column j is dominated by column k. Note that a payoff matrix is constructed such that the payoffs are from the perspective of Player I or row player, so that negative values correspond to positive payoffs for Player II or column player. A row is dominated by another row if the entries in the dominated row are less than or equal to the entries in the dominating row. Dominated strategies extend to probability-weighted combinations of columns and rows. For example, column j is dominated by a combination of columns k and l if

$$a_{ij} \geq s a_{ik} + t a_{il} \text{ for all } i, \text{ where } s + t = 1. \qquad (2)$$

When $n = 3$, the payoff matrix of basic Silverman's game cannot be reduced by domination, and we obtain $p = [¼, ½, ¼]$, i.e. an optimal strategy is to choose 1 with probability ¼, 2 with probability ½ and 3 with probability ¼. When $n = 4$, column 3 is dominated by column 4, and it follows that row 3 is dominated by row 4. From the resulting 3 × 3 matrix, the solution is $p = [¼, ½, 0, ¼]$. By induction on $n \geq 5$, it is easy to show using dominated strategies that columns 3 and 4 are dominated by column 5. Moreover, columns 6 to n are dominated by column 1. The result is a 3 × 3 payoff matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 5 \end{array} \\ \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \end{array} \qquad (3)$$

The payoff is the entry in the matrix corresponding to the row and column of the choices of the two players. For example, if row player chooses 5 and column player chooses 1, because $5 \geq 3 \times 1$, the corresponding entry of the matrix in (3) is a payoff of -2, i.e. column player wins 2 from row player.

Linear programming is useful in solving payoff matrices. Unfortunately, automated solvers arrive at a particular solution with no indication whether the obtained solution is unique or not. Analysis based on basic principles may provide a better solution. Let $p = [p_1, p_2, p_5]$ be an optimal mixed strategy for row player, where the $p_i \geq 0$ ($p_3 = p_4 = 0$) and

$$\textstyle\sum_i p_i = 1 \qquad (4)$$

Minimax strategy (see Ferguson, 2000, for example) requires that

$$\begin{aligned} p_2 - 2p_5 &\geq v \\ -p_1 + p_5 &\geq v \\ 2p_1 - p_2 &\geq v \end{aligned} \qquad (5)$$

Since the game is fair then $v = 0$. Combining the second and third equations in (5), we get

$$-p_2 + 2p_5 \geq 0 \qquad (6)$$

Comparing (6) and the first equation in (5), then the inequality must be an equality. In fact, by combining the equations two at a time, the inequalities in (5) all turn out to be equalities. The probabilities can be solved using the constraint (4) to get $p = [¼, ½, ¼]$. Thus, for any n, the optimal strategy is $p = [¼, ½, 0, 0, ¼, 0, 0, …]$. In other words, a player must select 1 25% of the time, 2 50% of the time and 5 25% of the time. This result was also proved by Evans and Heuer in the general case (1992).

Variation of Silverman's Game

In this variation of the game, the two players choose integers between 1 to n simultaneously as before. The player who chooses the larger number wins the difference between the numbers, except when the larger number is three times or more than the smaller number, in which case the payoff to the player selecting the smaller number is twice the difference between the numbers. This game, just as in the basic version, is symmetric and fair, with value $v = 0$, and both players have the same optimal strategies.

The solution when $n = 2$ is trivial: each player simply chooses 2, which is a saddle point of the payoff matrix. For $n = 3$, the payoff matrix cannot be reduced by domination. The solution is obtained as $p = [1/6, 2/3, 1/6]$. For $n = 4$, the payoff matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & -1 & 4 & 6 \\ 1 & 0 & -1 & -2 \\ -4 & 1 & 0 & -1 \\ -6 & 2 & 1 & 0 \end{pmatrix} \end{array} \qquad (7)$$

Using dominated strategies, note that 3 is dominated by using a strategy of choosing 2 50% of the time and choosing 4 50% of the time. The resulting 3 × 3 matrix is easily solved to get $p = [2/9, 2/3, 0, 1/9]$. For $n = 5$, the payoff matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & -1 & 4 & 6 & 8 \\ 1 & 0 & -1 & -2 & -3 \\ -4 & 1 & 0 & -1 & -2 \\ -6 & 2 & 1 & 0 & -1 \\ -8 & 3 & 2 & 1 & 0 \end{pmatrix} \end{array} \qquad (8)$$

Note that choosing 3 50% of the time and 5 50% of the time dominates 4. Likewise, choosing 5 50% of the time and 2 50% of the time dominates 3. This reduces the system to the 3 × 3 matrix below

$$\begin{array}{c} \\ 1 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 5 \end{array} \\ \begin{pmatrix} 0 & -1 & 8 \\ 1 & 0 & -1 \\ -8 & 1 & 0 \end{pmatrix} \end{array} \qquad (9)$$

The solution to this is

$$p_1 = 1/4, p_2 = 2/3, p_5 = 1/12 \qquad (10)$$

For $n = 6$, the payoff matrix is as follows:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & -1 & 4 & 6 & 8 & 10 \\ 1 & 0 & -1 & -2 & -3 & 8 \\ -4 & 1 & 0 & -1 & -2 & -3 \\ -6 & 2 & 1 & 0 & -1 & -2 \\ -8 & 3 & 2 & 1 & 0 & -1 \\ -10 & -8 & 3 & 2 & 1 & 0 \end{pmatrix} \end{array} \qquad (11)$$

Choosing 1 dominates choosing 6, hence using the same procedure, the payoff matrix is reduced to (9). For $n = 7$, note that selecting 7 is dominated by selecting 1, and as before, the system reduces to (9). For $n = 8$, following the previous solutions, choices 3, 4, 6 and 7 are eliminated by domination. Choosing 8 is then dominated by choosing 1 and again the system is reduced to (9), with solution (10). From here on, the nontrivial cases are when $n = 11$ and 14, hence these will be examined in detail.

When $n = 11$, choices 6, 7, 9, 10 are all eliminated by simple domination. Then choosing 4 is dominated by a mixed strategy of choosing 3 50% of the time and choosing 5 50% of the time. This reduces the matrix to

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 5 & 8 & 11 \end{array} \\ \begin{pmatrix} 0 & -1 & 4 & 8 & 14 & 20 \\ 1 & 0 & -1 & -3 & 12 & 18 \\ -4 & 1 & 0 & -2 & -5 & 16 \\ -8 & 3 & 2 & 0 & -3 & -7 \\ -14 & -12 & 5 & 3 & 0 & -3 \\ -20 & -18 & 16 & 7 & 3 & 0 \end{pmatrix} \end{array} \qquad (12)$$

Now choosing 11 is dominated by choosing 1. After eliminating 11, 3 is dominated by choosing 2 50% of the time and choosing 5 50% of the time. Subsequently, choosing 8 is dominated by choosing 1, hence the matrix is reduced to (9) as before.

Next consider $n = 14$. As before, we can eliminate 6, 7, 9, 10, 12, 13 using simple domination. The payoff matrix reduces to

$$\begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 8 & 11 & 14 \\ \hline 1 & 0 & -1 & 4 & 6 & 8 & 14 & 20 & 26 \\ 2 & 1 & 0 & -1 & -2 & -3 & 12 & 18 & 24 \\ 3 & -4 & 1 & 0 & -1 & -2 & -5 & 16 & 22 \\ 4 & -6 & 2 & 1 & 0 & -1 & -4 & -7 & 20 \\ 5 & -8 & 3 & 2 & 1 & 0 & -3 & -6 & -9 \\ 8 & -14 & -12 & 5 & 4 & 3 & 0 & -3 & -6 \\ 11 & -20 & -18 & -16 & 7 & 6 & 3 & 0 & -3 \\ 14 & -26 & -24 & -22 & -20 & 9 & 6 & 3 & 0 \end{array} \qquad (13)$$

This matrix cannot be reduced by domination, although one may notice that 8, 11 and 14 are nearly dominated by 1. Imagine that the game is of perfect information for column player, in other words, column player can foresee what strategy row player chooses, and counter-acts accordingly. The optimal strategy for column player can be assessed using best response to row player's pure strategies. If row player selects 1, column player's best response is to select 2. If row player selects 2, column player's best response is 14 for a payoff of 9, while 1 would be a next best response providing a payoff of 8. Thus with perfect information, it seems reasonable that 8, 11 and 14 are viable strategic choices. In the absence of perfect information, this is not the case.

To solve the system, we set up the 8 Minimax equations just as was done in (5), and then combine the inequalities. An equivalent and less tedious method is to let A be the matrix in (13), then perform row operations on A^T involving the addition of positive multiples of one row to another row. The sign of the multiplier ensures the inequalities do not change direction. Eliminating the first and second entries in the 3rd to 8th rows of A, the 5th row is reduced to the following inequality:

$$-6p_3 - 3p_4 - 135p_8 - 198p_{11} - 261p_{14} \geq 0 \qquad (14)$$

Clearly, this equation will only be satisfied when $p_3 = p_4 = p_8 = p_{11} = p_{14} = 0$. This implies that (13) reduces to (9), with an optimal solution given by (10). Moreover, to show that this strategy is stable, let the vector $p = [1/4 \;\; 2/3 \;\; 0 \;\; 0 \;\; 1/12 \;\; 0 \;\; 0 \;\; 0]$. Then if row player uses strategy p, row player's expected payoff is

$$p\mathrm{A} = [0 \;\; 0 \;\; 0.5 \;\; 0.25 \;\; 0 \;\; 11.25 \;\; 16.25 \;\; 21.75] \qquad (15)$$

Against this strategy, column player cannot choose 3, 4, 8, 11 and 14 with any frequency as this will result in a positive payoff for row player. Hence the best column player can do is to choose among 1, 2 and 5 for a zero payoff, and the optimal strategy for this is (10). If column player deviates from the prescribed probabilities of 1, 2 and 5, the strategy will be exploitable by a suitable counter strategy. Hence (10) is the unique solution to the game when $n = 14$.

Extending the game past $n = 14$ is trivial, because for each $k \geq 15$, k would be completely dominated by 1. This is clear, because k loses against 1, 2 and 5, whereas 1 ties with 1, loses against 2 (with a loss of 1 instead of $2(k - 1)$), and wins against 5. The matrix again reduces to (9). This proves that the unique optimal strategy for this modified Silverman's game for $n \geq 5$ is given by (10).

It has already been shown that the solution is different when one player (say, column player) has perfect information and can anticipate the other player's move. Assuming that row player is playing the optimal strategy (10), then column player will be choosing between best responses of 2, 5 and 14 accordingly. However, if the information is not perfect with a margin of error, it turns out that column player must be at least 95.97% certain that row player is choosing 5 just to breakeven by choosing 14 instead of 1. Thus even with near-perfect information at 5% error, the game theoretical optimal strategy (10) still wins.

CONCLUDING REMARKS

The basic Silverman's game wherein the payoffs are 1 (if the larger number wins) and 2 (if the smaller number wins by being less than or equal to 1/3 of the larger number) and the variation with varying payoffs have solutions which reduce each player's choices to the set {1, 2, 3} when $n = 3$, {1, 2, 4} when $n = 4$, and {1, 2, 5} when $n \geq 5$. In the latter case, the strategy set can be summarized as follows: 2 is the only choice that beats 1, 5 is the choice that beats 2 maximally, while 1 wins against all numbers from 3 up including 5. Modifying the payoff scheme wherein the winner is paid the difference between the numbers if the larger number wins or twice the difference if the smaller number wins affects the frequency with which the players should play these three choices. In the variation of the game, 2 should be played more often, 2/3 as opposed to ½ in the original form, while 5 should be played less often, 1/12 instead of 1/4. 1 is played with the same frequency of ¼ in both versions. The variation of Silverman's game could similarly be solved when the multiplier is changed, for example the larger number wins if it is less than 4 times the smaller number.

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CONFLICTS OF INTEREST

There are no conflicts of interest.

REFERENCES:

De Asis R. Game theory solution to Silverman's game. Undergraduate Research Paper, University of the Philippines 2012.

Evans RJ. Silverman's game on intervals. Amer. Math. Monthly 1979; 86: 277-281.

Evans RJ, Heuer GA. Silverman's game on discrete sets. Linear Algebra and its Applications 1992; 166: 217-235.

Ferguson T. Two player zero-sum games. In: Game Theory. UCLA Electronic Text, 2000; 30-31.

Heuer GA, Leopold-Wildburger U. Nonsingularity of some classes of matrices, and minimal solutions of Silverman's game on discrete sets. Mathematical Computational Modeling 1993; 18 (12): 1-31.

Heuer GA. Which four-part partition games are isomorphic to Silverman's game? European Journal of Operational Research 2002; 137: 588–592.