# When is the Pareto choice from a finite set invariant to variations in weight and values of multiple performance criteria?

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A simple procedure is described for selecting one option from a finite set or list of n options. Each option is associated with m performance criteria, features, or attributes. A weighted sum of the values of the normalized criteria is calculated for each option and the option with the largest weighted sum is selected. If the list contains Pareto-optimal options the winning option is Pareto optimal. It is shown that the winning option is invariant in spite of variations in the weights provided that these remain in a specified subspace of the multi-dimensional space of the weights that includes the chosen weights. Moreover, the winning option is invariant in spite of variations in the values of the multiple criteria provided that the variations remain in specified subspaces of the criteria space that includes the original criteria values. These invariance properties are reported for the first time in this paper. The entire process is called the invariance method. An upper bound for the maximum number of pairwise comparisons to identify all the *k* Pareto-optimal options in a finite set of n options, is derived. For each pair of *n* and *k*, where *n* is greater than or equal to k, there is a list such that the upper bound is reached.

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## INTRODUCTION

In applications of decision-making, a generic problem is to choose an option from a set of many options, possibly infinitely many, where each option is associated with multiple optimization indices or performance objective functions, or performance criteria. In this regard, the choice is not simple because for any option the associated performance criteria are not necessarily all better than those associated with other options (Zadeh 1963).

The concept of Pareto optimality (Pareto, 1896) is relevant for the finite set case also. An extensive literature exists for Pareto optimization, and as representative references see (Chankong and Haimes 1983, Ehrgott 2005, Eckenrode RT 1965, Fishburn PC 1967, Geoffrion AM 1968, Marler and Arora 2010, Sawaragi et al 1985, Rao 2007,) but the theory covers general cases where the space of choices is in a set of continuum of real numbers. In this paper, we consider the much simpler case where the set of choices is a finite set. In many applications, a simpler theory for choosing from a finite set is sufficient. We focus on the finite set case of the weighted sum method for Pareto-optimization (Fishburn AM 1967, Marler and Arora 2010, Rao 2007).

In this paper novel invariance properties of the weighted sum Pareto-optimization choice from a finite set in spite of variations, are described for the first time. The variations are in the changes in the weights and the values of multiple performance indices. To the best of our knowledge, no invariance properties have been reported for the case when options are from a continuum of real numbers. We provide a simple theory and procedure for selecting an option from a finite set of options, where each option is associated with multiple optimization criteria. This theory underlies almost all engineering design methods and comparison of systems with multiple performance criteria. The theory can be applied to many practical situations instead of the widely documented more complex theory for sets of continua of real numbers (Sawaragi et al 1985). The theory is vastly simpler even in comparison with combinatorial multiple criteria optimization such as bi-criteria integer programming (Ehrgott 2005).

## PARETO-OPTIMALITY OVER A FINITE SET

In this section we begin by providing illustrative examples of Pareto-optimality before considering the general case of a list of *n* options.

**Example 1.**

Consider a problem of selecting from several possible options. Each option is associated with multiple performance criteria or performance objective functions. For each option, the value of each performance criterion is a non-negative real number. Table 1 displays a list of four options, in the last four rows, labeled as Option 1, Option 2, Option 3, and Option 4, in Column 1. The second, third and fourth columns provide values of Performance Criteria 1, 2, and 3. As an illustrative example, suppose a student is in her last semester before graduating from one of the National Science High Schools in the Philippines, and she wants to pursue a college degree in engineering or information technology. After talking to many friends already in college, studying web sites of various colleges and universities, and visiting many campuses with her parents, she concludes that she could see herself studying at one of four higher education institutions (HEI) based on what see saw, her gut feelings, and her acceptance at the four schools. Her next step is to choose among the four. She decided that she would use three criteria. The first performance criterion (PC1) is cost of education. She estimated the total cost for obtaining a degree and the cost in Philippine pesos is indicated in Column 2. These are illustrative and do not indicate actual data. Table 1 indicates that HEI4 is the least expensive and HEI2 is the most expensive. There were HEi's she visited that have lower cost but they were not acceptable to her based on her gut feelings. Later, we will describe a procedure for transforming the data so that the highest numeral value corresponds to the smallest cost. The second Performance Criterion she wants to use is Quality of Education, as indicated by external accreditation by the Accreditation Board for Engineering and Technology, or ABET (based in the USA, the international standard in engineering and computing education). She counted the number of programs accredited by ABET with the view that the one with the most number is the most desirable. Her data are indicated under the Criterion 2 column, showing HEI4 as best and HEI1 as worst of the four. For a third criterion she wanted to compare the schools based on effectiveness of the career center, placement centers, and satisfaction of employers with the schools. Her personal scores for the 4 HEIs are indicated in the last column of Table 1, where option 4 is best.

**Table 1: Option 4 is a dominant option because for each of the three performance criteria it is better than the other three options.**

| Option number | Cost of education | Number of ABET accredited programs | Effectiveness of career support |
|---|---|---|---|
| 1 | Php 520,000.00 | 6 | 3 |
| 2 | Php 576,000.00 | 7 | 2 |
| 3 | Php 473,000.00 | 18 | 1 |
| 4 | Php 350,000.00 | 20 | 4 |

We perform pairwise comparison of the rows. Comparing Option 4 with Option 3, we note that performance criteria 1, 2, and 3 are worse for Option 3 compared to Option 4 and we say that Option 3 is inferior to or dominated by Option 4, and conclude that Option 3 will never be selected in the end because Option 4 is better. Similarly, Option 2 is inferior to Option 4 and Option 1 is inferior to Option 4. After three pairwise comparisons we conclude that Option 4 dominates all other options, and it is better than the other options based on each of the three criteria.

**Example 2.**

In Example 2, consider an illustrative challenge of choosing a smart phone to purchase. The latest versions from various vendors are amazing and they are significantly better than last year's models. After scanning the Internet and studying various reviews, we preselect four brands and models of smart phones. Please refer to Table 2. For performance criterion 1 we choose cost, ranging from Php. 25,000.00 to Php. 31,000.00, as indicated in Table 2. Later we will describe a procedure whereby the highest numerical value corresponds to the least cost. For performance criterion 2 we choose the quality/sharpness of the images from the camera, giving a grade of 4 for best and 1 for worst. Finally for performance criterion 3 we choose ergonomics of the smart phone including ease of use, placement of ID biosensors, etc., with 4 as best and 1 as worst. The data are indicated in Table 2. We perform pairwise comparisons starting from the bottom. Comparing Options 4 and 3 we note that neither one dominates the other. Compare Options 4 and 2, we note that neither Option dominates the other. Next compare Options 4 and 1, and note that neither Option dominates the other. Next compare Option 3 with Option 2 and we note that Option 3 dominates Option 2. Finally we compare Option 3 with Option 1 and we note that Option 3 dominates Option 1. We are left with Options 3 and 4. Although performance criteria 1 and 3 are better for Option 4, performance criterion 2 is better for Option 3. Options 3 and 4 are examples of options that are said to be Pareto-optimal. In this example we performed 5 pairwise comparisons to conclude that there are two dominated options and two Pareto-optimal options. At this stage we eliminate options 2 and 1. Later we will discuss how we select from the remaining 2 Pareto-optimal options.

**Table 2: Option 3 dominates Options 1 and 2. Options 3 and 4 are Pareto-optimal options.**

| Option number | Cost of cell phone | Camera image sharpness | Ergonomics |
|---|---|---|---|
| 1 | Php 30,000.00 | 2 | 1 |
| 2 | Php 31,000.00 | 3 | 2 |
| 3 | Php 27,000.00 | 4 | 3 |
| 4 | Php 25,000.00 | 1 | 4 |

**Definition of dominant option**

In a list of *n* options, an option *i* is a dominant option if for any other option *j* the features of option *i* are all better than those for option *j*.

(It is possible to broaden this definition as follows: an option *i* a dominant option if for any other option *j* the features of option *i* are all better than or equal to those for option *j* and that there is at least one feature where option *i* is strictly better than option *j*. For simplicity we will adopt the original definition).

**Definition of dominated or inferior option**

In a list of *n* options, an option *i* is dominated by (or inferior to) option *j* if all the features of option *j* are better than the corresponding features of option *i*.

As in the dominant option, this could be generalized somewhat but we will use the first definition.

**Definition of Pareto-optimal option**

In a list of *n* options where there are no dominant options, an option i is said to be Pareto-optimal if there is no option that dominates option i and that for any option *j* such that option *j* has a better feature than option *i*, option *j* has another feature where it is worse than that for option *i*. Pareto-optimality is also known as Pareto efficiency.

In a list where there is a dominant option, clearly there is no other option that has a feature that is better than the corresponding feature of the dominant option.

**Example 3.**

In Example 3, we consider the data in Table 3. In all three performance criteria, the higher the numerical rating the more desirable it is. We perform pairwise comparisons starting from the bottom. Comparing Options 4 and 3 we note that neither one dominates the other. Compare Options 4 and 2, we note that neither one dominates the other. Compare Options 4 and 1, and note that Option 4 dominates Option 1, so Option 1 is eliminated. Next we compare Options 3 and 2 and note that neither one dominates the other. Options 2, 3, and 4 remain. Options 2, 3, and 4 are said to be Pareto-optimal. In this example we performed 4 pairwise comparisons to conclude that there is one dominated option and three Pareto-optimal options. Later we will discuss how we select from the remaining three Pareto-optimal options.

**Table 3: Option 4 dominates Option 1. Options 2, 3, and 4 are Pareto-optimal.**

| Option number | Performance Criterion 1 | Performance Criterion 2 | Performance Criterion 3 |
|---|---|---|---|
| 1 | 0.8 | 0 | 0.6 |
| 2 | 0 | 0.8 | 0.4 |
| 3 | 0.5 | 1 | 0 |
| 4 | 1 | 0.7 | 1 |

**Example 4.**

In Example 4 consider the data in Table 4. The higher the numerical values of the performance criteria rating, the more desirable the option is. As in the previous examples we perform pairwise comparisons starting from the bottom. After six pairwise comparisons we conclude that all four options are Pareto-optimal.

**Table 4: All options are Pareto-optimal.**

| Option number | Performance Criterion 1 | Performance Criterion 2 | Performance Criterion 3 |
|---|---|---|---|
| 1 | 1 | 0 | 0.6 |
| 2 | 0 | 1 | 0.4 |
| 3 | 0.5 | 0.7 | 0 |
| 4 | 0.7 | 0.5 | 1 |

In the previous examples we observed that even for the same number of options, the number of pairwise comparisons might differ from case to case. Given a finite list of n options, we are interested in identifying the Pareto-optimal options. It is possible that all n options are Pareto optimal, or perhaps only a subset, if any, might be Pareto-optimal. Let us now consider the general case of a finite list of n options, where each option is associated with m performance criteria, and n and m are positive integers both greater than 1. Perform needed pairwise comparisons. If no inferior or dominated options were found, clearly there would be n Pareto-optimal options. It is possible that n – 1 options could be found to be inferior, in which case one option would dominate all other options and there would be no Pareto-optimal option. Let us provide upper bounds on the number of needed pairwise comparison to determine the Pareto-optimal set.

## PAIRWISE COMPARISONS IN A FINITE SET OR LIST OF n OPTIONS

As indicated in the previous examples, we need to perform pairwise comparisons and when n is large, the number of pairwise comparisons could be very large. The smallest allowed value of *n* is 2 and in this case there is only one pairwise comparison. Theorem 1 provides a formula for the number of different comparisons in a list of *n* options.

**Theorem 1**

Denote the maximum number of different pairwise comparisons for a list of *n* options where $n > 1$, by $q(n)$. Then

$$q(n) = \frac{n(n-1)}{2} \tag{1}$$

This is proved in Appendix A.1.

The actual number of different pairwise comparisons needed to identify Pareto-optimal options may be less. For example for $n = 4$, $q(4) = 6$, but in Example 1, where $n = 4$, there are only *3* comparisons needed. In Example 4 where $n = 4$, we need to make 6 comparisons. Theorem 1 gives an upper bound on the number of pairwise comparisons to identify all Pareto-optimal options. When all options are Pareto-optimal, it is possible that the number of pairwise comparisons that are needed could reach the bound in Theorem 1. However, when there are Pareto-optimal options (at least 2) Theorem 2 provides a lower upper bound.

**Theorem 2**

Denote by *p(n)*, the maximum number of different pairwise comparisons for a list of *n* options, containing k Pareto-optimal options, where n and k are integers, $2 \le k \le n$.Then *p(n)* is bounded from above by *q(n, k)*

$$p(n) \le q(n,k) = \frac{k(2n-k-1)}{2}$$
$$2 \le k \le n. \tag{2}$$

This is proved in Appendix A.2. When all options are Pareto-optimal, the upper bound on the number of iterations is the same as that given by Theorem 1, so that the number of pairwise comparisons is of Order $n^2$. On the other hand if $k = 2$, Equation (2) reduces to $q(n, 2) = (2n - 3)$ and the number of pairwise comparisons is of Order *n*.

## SCALING, LINEAR SHIFT, AND NORMALIZATION

In Examples 1 and 2, Performance Criterion 1 is cost in Philippine pesos whereas the other criteria have other dimensional units or dimensionless. In applications, it is usually meaningful to rank order the importance of the various performance criteria. Moreover, it is meaningful to assign percent (%) importance to each criterion. It is customary to form a weighted linear combination of the numerical values of the criteria. When the criteria have different dimensions, it does not make sense to add them directly, so we need to normalize to convert all to be dimensionless. In Examples 1 and 2, smaller cost is desirable but it is more convenient if the criteria could be transformed so that larger is better. Finally, it is desirable to convert all of the numerical scores such that on a scale of 1 to 10, 10 is best and 1 is worst. In applications where the raw numerical scores are supposed to be most desirable when the numerical score is lowest such as in Examples 1 and 2 for Performance criterion 1 we propose the following scaling, linear shift, and normalization:

$$PC_{norm} = 9\frac{Max_{raw} - PC_{raw}}{Max_{raw} - Min_{raw}} + 1 \tag{3}$$

The scaling, linear shift, and normalization using Equation (3) preserve the relative rank ordering of values for each performance criterion.

In Example 1, Table 1, the maximum and minimum values for raw Performance Criterion 1 are 576,000 and 350,000 so that the normalized values for Performance Criterion 1, as shown in Column 2 of Table 5 are: Option 1, *3.23*; Option 2, *1.00*; Option 3, *5.10*, Option 4, *10.00*.

When raw high numerical values are associated with more desirable values, we propose the scaling, linear shift, and normalization formula:

$$PC_{norm} = 9\frac{PC_{raw} - Min_{raw}}{Max_{raw} - Min_{raw}} + 1 \tag{4}$$

The scaling, linear shift, and normalization using Equation (4) preserve the relative rank ordering of values for each performance criterion.

**Table 5: The data in this table are the scaled, shifted and normalized data of Table 1 using equations (3) and (4).**

| Option number | Performance Criterion 1 | Performance Criterion 2 | Performance Criterion 3 |
|---|---|---|---|
| 1 | 3.23 | 1 | 7.00 |
| 2 | 1.00 | 1.64 | 4.00 |
| 3 | 5.10 | 8.71 | 1.00 |
| 4 | 10.00 | 10.00 | 10.00 |

**Example 5.**

In Example 1, Table 1 for raw values for Performance Criterion 2 as indicated in column 3, the normalized values are: Option 1, 1; Option 2, 1.64; Option 3, 8.71; Option 4, 10.00, as shown in Column 3 of Table 5. The complete normalization of the data in Table 1 is shown in Table 5. The relative rank ordering of the values for each Performance Criterion is preserved. Note that the pairwise comparison of rows in Table 5 will result in the same conclusion as the one in Example 1, whereby Option 4 dominates all the other three options.

**Example 6**.

Normalizing the numerical values in Table 2 results in Table 6. As in Table 2, Options 1 and 2 are inferior and there are two Pareto-optimal options. The result is the same as for Table 2.

**Table 6: This is the scaled, shifted, and normalized version of Table 2 using equations (3) and (4).**

| Option number | Performance Criterion 1 | Performance Criterion 2 | Performance Criterion 3 |
|---|---|---|---|
| 1 | 2.50 | 4.00 | 1.00 |
| 2 | 1.00 | 7.00 | 4.00 |
| 3 | 10.00 | 1.00 | 10.00 |
| 4 | 7.00 | 10.00 | 7.00 |

**Example 7.**

Table 7 shows the normalization of the data in Table 4. Note that as in Table 4, the four options are all Pareto-optimal. The result is the same as that for Table 4.

Unless noted otherwise, all subsequent Performance Criteria values in the paper are scaled, shifted, and normalized values.

## WEIGHTED SUM OF PERFORMANCE CRITERIA

When given a finite list of options, each with multi-criteria features, usually there are Pareto-optimal options. In applications, it is important and useful to have a selection procedure to choose one option among the Pareto-optimal options. In this section we describe a popular procedure for selection, the weighted sum of performance criteria (Ehrgott 2005, Marner and Arora 2010).

**Example 8.**

Consider the normalized list in Table 6. Suppose the four options represent four graduate students in a small class. Performance Criterion 1 represents the normalized grades in the final examination. Performance Criterion 2 represents the normalized grades for the mid-term examination, and Performance Criterion 3 represents the normalized grades for the homework assignments. Suppose that the teacher assigns 50% weight to the final examination, 30% to the mid-term examination, and 20% to the homework assignments. We multiply the weight for each of the criteria by the normalized value of the criteria and add the three products for each student. The total composite scores are Student 4, 7.9; Student 3, 7.3; Student 2, 3.4; Student 1, 2.63. Student 4 attains the maximum total composite score. Note that Student 4 is one of the Pareto-optimal options. What if the teacher were to consider a different weight distribution, such as 70% for the final examination, 10% for the mid-term examination and 20% for the homework assignments. Then the total composite scores would be: Student 4, 7.3; Student 3, 9.1; Student 2, 2.2; Student 1, 2.35. In this case Student 3 would be at the top of the class. Note that Student 3 is a Pareto-optimal option. In this example for a given set of weights, the maximum total composite score corresponds to the top student in the class, and the class rankings for the other students correspond to their total composite scores.

**Theorem 3**

Given any finite list of *n* options, each with *m* Performance Criteria, where *m* and *n* are integers, and where there are at least 2 Pareto-optimal options.Denote the value of the *j*th Performance Criterion for option *i* by $PC_j^i$ and denote the weighted sum $\sum_{j=1}^{m} x_j\, PC_j^i$ by $WPC^i$

$$WPC^i = \sum_{j=1}^{m} x_j\, PC_j^i \tag{5}$$

Where $x_j \geq 0$ and $\sum_{j=1}^{m} x_j PC_j^i = 1$. Denote the value of $i$ that results in $\max_i [WPC^i, i = 1,2, \ldots, n]$ by $i$*. Then for each choice of the set $\{x_j\}_1^m$, $i$* is a Pareto-optimal option.

Proof is in Appendix A.3. In applications, there is merit in assigning relative importance to the Performance Criteria, by assigning weights as percent/100, taking a weighted linear combination, and maximizing the weighted sum with respect to the options. This process yields a choice that is a Pareto-optimal option, as stated in Theorem 3.

When the finite set of options has no Pareto-optimal option, then there must be a dominant option. In this case, regardless of the choice of weights, the maximizing value of $i$ will correspond to the dominant option.

In the theory for continuum sets, the maximum of the weighted sum over the infinitely many in the continuum set of options is a Pareto optimal option. When the options are concave functions, all Pareto-optimal points are obtained through the maximization of the weighted sums. If the functions are not concave, not all Pareto-optimal points can be obtained (Sawaragi et al 1985). This is similar to the case for combinatorial multicriteria optimization (Ehrgott 2005) and to the finite set case of this paper.

**Example 9.**

For the data in Table 7, form $WPC^i = x_1 PC_1^i + x_2 PC_2^i + x_3 PC_3^i$ where $x_j \geq 0, j = 1,2,3, x_1 + x_2 + x_3 = 1$. $PC_j^i$ is Performance Criterion $j$ for option $i$, where $j = 1, 2, 3$ and $i = 1, 2, 3, 4$.

$$\max_i [10x_1 + x_2 + 6.4(1 - x_1 - x_2) \text{ for } i = 1; x_1 + 10x_2 + 4(1 - x_1 - x_2) \text{ for } i = 2; 5.5x_1 + 7.3x_2 + (1 - x_1 - x_2) \text{ for } i = 3; 7.3x_1 + 5.5x_2 + 10(1 - x_1 - x_2) \text{ for } i = 4 \tag{6}$$

In accordance with Theorem 3, no matter what the values of the weights are, provided that they are nonnegative and add up to 1.00, the maximizing value of i is Pareto-optimal.

**Table 7: This table is a scaled, shifted, and normalized version of Table 4 using equations (3) and (4).**

| Option number | Performance Criterion 1 | Performance Criterion 2 | Performance Criterion 3 |
|---|---|---|---|
| 1 | 10.00 | 1.00 | 6.40 |
| 2 | 1.00 | 10.00 | 4.60 |
| 3 | 5.50 | 7.30 | 1.00 |
| 4 | 7.30 | 5.50 | 10.00 |

**Example 10.**

In this example we show that it is possible that not all Pareto-optimal options can be obtained by maximizing the weighted sum in Theorem 2 no matter what we choose for the values of the weights. We will show that the Pareto-optimal option 3 in Table 8 cannot be obtained by maximizing the weighted sum in Theorem 3. Suppose that $i^* = 3$ is the maximizing value of $i$. Then there should exist weights $x_1, x_2, x_3$ such that

$$2x_1 + 2x_2 + 2(1 - x_1 - x_2) > 10x_1 + 10_{x2} + (1 - x_1 - x_2) \tag{7}$$

and

$$2x_1 + 2x_2 + 2(1 - x_1 - x_2) > x_1 + x_2 + 10(1 - x_1 - x_2) \tag{8}$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$, and

$$x_1 + x_2 + x_3 = 1 \tag{9}$$

Simplifying the inequality in (7) we obtain

$$x_1 + x_2 < \frac{1}{9} \tag{10}$$

Simplifying the inequality in (8) we obtain

$$x_1 + x_2 > \frac{8}{9} \tag{11}$$

The half spaces associated with the inequalities in (10) and (11), have no intersection. The nonnegative restrictions on the weights, the line associated with Equation (9), and the two half spaces have no intersection. Hence there is no set of weights such that i* = 3.

This example shows that maximizing the weighted sum in Theorem 3, for all possible weights, will not necessarily yield all the Pareto-optimal options. Thus this example shows that the converse of Theorem 3 is not true.

**Table 8: Comparing three new varieties of an edible plant. Example where option i = 3 is Pareto-optimal although it does not maximize the value of a weighted sum. Thus option 3 will not be chosen by the weighted sum method.**

| Option number | Resistance to pests | Least need for water | Tastiness |
|---|---|---|---|
| 1 | 10.00 | 10.00 | 1.00 |
| 2 | 1.00 | 1.00 | 10.00 |
| 3 | 2.00 | 2.00 | 2.00 |

To provide a specific context for this example, suppose that plant breeders were able to identify three new varieties of a specific edible plant and they would like to choose which one to provide farmers to replace the variety of the plant they currently grow. Performance criterion 1 is resistance to pests. Criterion 2 is amount of needed water. Criterion 3 is tastiness. Assigning weights to the three criteria makes sense. Using the method of weighted sums, the calculations demonstrate that Plant variety 3 will not be chosen. Depending on what weights are used, they would recommend Plant variety 1, which is best for resistance to pests and best for amount of water needed, or plant variety 2, which is best for tastiness. Although Plant variety 3 is Pareto-optimal, it is not best for any of the three criteria.

For a list that contains Pareto-optimal options, if an option contains the highest score for a performance criterion, there is a weight such that it is a winning option and hence it is Pareto-optimal. This is clear because a weight of 1.0 for that performance criterion and 0.00 for all the other criteria would result in that option receiving the highest value for the weighted sum.

For other Pareto-optimal options that do not contain the highest value for any criterion, there is no guarantee that maximizing the weighted sum would lead to that option, for any weights, as shown by Example 10 as a counter example.

## INVARIANCE OF THE WINNING PARETO-OPTIMAL OPTION TO WEIGHT VARIATIONS AND SENSITIVITY ANALYSIS

From Theorem 3, maximizing the weighted sum of performance criteria, leads to a Pareto-optimal option. How sensitive is the resulting winning Pareto-optimal option to the choice of the values of the weights? It is prudent to perform a sensitivity analysis on the effect of changes in the values of the weights (Cruz 1973).

For any weight in the (m-1)-dimensional hyper triangle subspace described in Theorem 3, the winning option corresponds to one of the k Pareto-optimal options. When the weight is changed slightly, it is natural to expect that the winning option might change to a different Pareto-optimal option. As we explore different weights, we could be reaching any of the other Pareto-optimal options. It was shown earlier that it might not be possible to reach all k Pareto optimal options. Theorem 3 guarantees that at least one of the Pareto-optimal options can be reached, but k provides an upper bound.

Although there are infinitely many points in the hyper triangle only at most k Pareto-optimal options can be associated with winning options. Suppose we choose a specific weight in the hyper triangle. The procedure in Theorem 3 will lead to a specific Pareto optimal option, which we label as $i^*$. Is it possible that for other weights near the chosen weights, the winning option might remain as $i^*$? Robustness property of this choice of weights is provided in Theorem 4.

**Theorem 4.**

For a specific weight selected from the (m – 1) –dimensional hyper triangle in Theorem 3, leading to a Pareto–optimal option $i^*$, there is a continuum of (m-1)-dimensional subspace of the hyper triangle, which includes the selected weight, such that for any weight in the subspace, the winning option remains as $i^*$.

See Appendix A.4 for a proof of the Theorem. Every choice of weights is robust or insensitive because there is a continuum of weight values that includes the selected weight, leading to the same Pareto-optimal choice. To distinguish this type of insensitivity from small values of sensitivity, we will refer to this as **invariance of the winning Pareto-optimal option to weight variations.**

In the case of infinitely many options in a continuum, any change in the weight would generally yield a different Pareto-optimal point. **The authors are not aware that there is a counterpart of Theorem 4 that provides invariance for the continuum case.**

**Example 11.**

Consider the data in Table 6. We know that Options 3 and 4 are Pareto-optimal. Let us determine the values of $x_1, x_2$, and $x_3$so that maximizing the weighted sum gives Option 3 as the winning Pareto-optimal choice. So

$$10x_1 + x_2 + 10(1 - x_1 - x_2) > 7x_1 + 10x_2 + 7(1 - x_1 - x_2) \tag{12}$$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ and

$$x_1 + x_2 + x_3 = 1 \tag{13}$$

Simplifying (12) yields

$$-12x_2 > -3 \text{ or } x_2 < \frac{1}{4} \tag{14}$$

The intersection of all the above conditions yields a non-empty subspace. This example shows that maximizing a weighted sum of the performance criteria yields Option 3 as the Pareto-optimal choice no matter what the values of the weights are, provided the weights are inside a portion of a triangle in two-dimensional subspace in the three dimensional space of the three weights. For option 4 to be the winning option, the inequality in (12) needs to be reversed leading to the reversal of the inequality in (14). Thus the triangle in 3-dimensional space is divided into two parts. In one part, the winning option is option 3 and in the other part, the winning option is option 4.

**Example 12.**

Consider the data in Table 9. Clearly all options are Pareto-optimal. Option 3 does not have any performance criterion that has the highest value among the options. Let us investigate whether Option 3 can be the winning option in maximizing the weighted sum. In order for Option 3 to be the winning option, there should exist weights such that

$$9x_1 + 9x_2 + 9(1 - x_1 - x_2) > 10x_1 + 10x_2 + (1 - x_1 - x_2) \tag{15}$$

and

$$9x_1 + 9x_2 + 9(1 - x_1 - x_2) > x_1 + x_2 + 10(1 - x_1 - x_2) \tag{16}$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ and

$$x_1 + x_2 + x_3 = 1 \tag{17}$$

Simplifying (15) and (16)

$$x_1 + x_2 \geq \frac{1}{9} \tag{18}$$

$$x_1 + x_2 \leq \frac{8}{9} \tag{19}$$

Thus the intersection of $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$, (17), and (20)

$$\frac{1}{9} \leq x_1 + x_2 \leq \frac{8}{9} \tag{20}$$

specifies the subspace such that any weight in the subspace would result in Option 3 as the winning option. This example shows a winning option that is not associated with the highest value for any performance criterion.

**Table 9: Example of a finite set of Pareto-optimal options, where the winning option (Option 3) in maximizing a weighted sum of criteria, does not have a criterion that has the highest value among options.**

| Option number | Performance Criterion 1 | Performance Criterion 2 | Performance Criterion 3 |
|---|---|---|---|
| 1 | 10.00 | 10.00 | 1.00 |
| 2 | 1.00 | 1.00 | 10.00 |
| 3 | 9.00 | 9.00 | 9.00 |

## INVARIANCE OF THE WINNING PARETO-OPTIMAL OPTION IN SPITE OF VARIATIONS IN PERFORMANCE CRITERIA VALUES

Heretofore, we assumed that the raw values of the Performance Criteria are known precisely. In practice there might be a variety of reasons why the values might be uncertain or subject to error tolerances. What might be the impact of these uncertainties on a possible change in the winning Pareto-optimal option when there are changes in the raw values of the Performance Criteria?

Consider the case when the winning option has a performance criterion value that is best among the options. Suppose that in spite of variations the highest raw value is still the highest raw value although the value may be different. Thus in the transformation and normalization the converted value would remain as 10.00. The normalized values for the other options might change but all are less than 10.00. The winning option would remain as before. Similarly, there could be variations in the other performance criteria, and simultaneous variations in all the performance criteria. As long as the criteria with the largest raw values remain with highest raw values, for the winning option, the corresponding transformed normalized values would still be 10.0. This invariance pertains to winning options that are associated with at least one performance criterion that is highest.

We shall refer to this absolute insensitivity as **invariance of the winning Pareto-optimal option to variations in the value of the performance criteria.**

**Example 13.**

Consider the data in Table 9. From Example 12, if we choose, $x_1 = 0.25, x_2 = 0.25, x_3 = 0.5$
the winning option will be i* = 3.

Suppose that for Performance Criterion 2, the raw values are uncertain or not accurately known for a variety of reasons, but for Option 1 suppose that we know that it is still the highest, and for Option 2, it is still the lowest. The transformed normalized values will still be 10 and 1 respectively. For Option 3, instead of the normalized value of 9, we will denote the normalized value to be y, where y is unknown but it is between 1.00 and 10.00.

Then inequality (15) becomes

$$9(0.25) + y(0.25) + 9(0.5) > 10(0.25) + 10(0.25) + 0.5 \quad (21)$$

or $0.25y > 5.5 - 6.75$. (22)

Inequality (16) becomes

$$9(0.25) + y(0.25) + 9(0.5) > 0.25 + 0.25 + 10(0.5) \quad (23)$$

or $0.25y > 5.5 - 6.75$, (24)

which is identical to (22). Clearly any value of y, where $1.00 < y < 10.00$ will not change the selection of Option 3 as the winning option. This example demonstrates the invariance of the winning option with respect to variations in the values of the raw values of a Performance Criterion, provided the best raw value is still the best raw value for the winning option even if the value is different.

When the winning option does not have a performance criterion value that is highest, such as in Example 12, the invariance property might not hold if the variation is substantial even if the relative rank order of the new value is preserved.

**Example 14.**

Suppose we have the situation in Table 9 and suppose that the weights are such that Option 3 is the winning option. If there is a variation in the values of the criteria so that the situation in Table 9 becomes that in Table 8, Option 3 will no longer be a winning option although the relative ranking for all three criteria remains.

However, if the change in the value of the criterion is sufficiently small, by continuity arguments, there should be a range of variations so that invariance is maintained.

**Example 15.**

Consider the data in Table 9. From Example 12, if we choose, $x_1 = 0.25, x_2 = 0.25, x_3 = 0.5$
the winning option will be i* = 3. Suppose that Criterion 1 has a variation such that the value for Option 1 is still the highest, the value for Option 2 is still the lowest, and the normalized new value for Option 3 is x. For Criterion 2 suppose that the variation is such that it is still highest for Option 1, still lowest for option 2 and the new value for Option 3 is y. Finally, suppose that for Criterion 3 the variations are such that for Option 1 it is still the lowest, for Option 2 it is still the highest, and for Option 3 the new value is z. Let us investigate what values of x, y, and z, would maintain the invariance of Option 3 as the winning option.

Inequality (21) becomes

$$0.25x + 0.25y + 0.5z > 10(0.25) + 10(0.25) + 0.5 = 5.5 \quad (25)$$

Inequality (23) becomes

$$0.25x + 0.25y + 0.5z > 0.25 + 0.25 + 10(0.5) = 5.5 \quad (26)$$

which is identical to Inequality (25). In addition

$$1 < x < 10 \quad (27)$$

$$1 < y < 10 \quad (28)$$

$$1 < z < 10 \quad (29)$$

The intersection of inequalities (26), (27), (28), and (29) is not empty, and it defines the subspace for x, y, and z that preserves the invariance of the winning Option 3. For example it contains x = 9, y = 9, and z = 9, the original values of the three criteria. It contains the subspace: $5.5 < x < 10$, $5.5 < y < 10$, $5.5 < z < 10$, but the allowed subspace is larger than this. Thus the winning Option 3 is invariant to variations in all the three criteria provided that the new values for x, y, and z remain in the subspace described above. We also assumed that the highest and the lowest values of the three criteria remain with their options.

## SUMMARY

In this paper, we considered the problem of selecting an option from a finite set or list of n options. Each option is associated with m performance criteria or features. The features are scaled, shifted, and normalized on a scale of 1 to 10 such that 10 is best and 1 is worst. A weighted sum of the criteria is calculated for each option and the option with the largest weighted sum is selected. The winning option is shown to be a Pareto optimal option. Most importantly, it is shown that the winning option is invariant to variations in weights in the sense that there is a continuum of real number weights in a subspace of the multi-dimensional space of the weights that includes the chosen weight, such that the winning option remains the same for all weights in the subspace. There is no counterpart of this invariance when the options are infinitely many in a continuum. Furthermore, the winning Pareto-optimal option remains invariant with respect to variations in the raw values of the performance criteria provided that the variations remain in specified subspaces.

We will call this entire process the **invariance method.**

It is shown through a counter example that it is not always possible to obtain all Pareto-optimal options by maximizing the weighted sum of the numerical values of the normalized criteria, no matter what the values of the weights are. This is analogous to the infinitely many options in a continuum when the functions are not concave.

Upper bounds for the maximum number of pairwise comparisons needed to identify all the p Pareto-optimal options in a list of n options, are derived. For each pair of n and p where n is greater than or equal to p, there is a list such that the upper bound is attained for the list.

## CONCLUSION

The problem of choosing an option from a finite set or list of Pareto-optimal options is vastly mathematically simpler, compared to the case where the set is not only infinite but also a continuum. In practical applications, for a variety of reasons, the choice is ultimately from a finite set. The process described in this paper, the **invariance method**, could be applied to the design process in engineering, computing, and other fields where there might be additional considerations such as compliance with standards and compliance with multiple constraints. These preliminary considerations should be accomplished before applying the procedures in this paper, so that all the options comply with the required standards and that options not in compliance with constraints have been eliminated already. The **invariance method**, could provide a theoretical foundation for design methodologies and design trade-offs, whenever there are a finite number of design options to be considered. The winning design option remains invariant with respect to (a) a range of weights, and (b) with respect to changes in values in the performance criteria, provided that the variations remain in specified subspaces. This provides additional confidence about the chosen design option. Also, the methodology described in this paper could be applied to the comparative evaluation of several systems or processes with multiple performance criteria associated with each of the systems/processes.

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## Appendix A.1

**Theorem 1**

Denote the maximum number of different pairwise comparisons for a finite set or list of *n* options where *n > 1*, by *q(n)*. Then

$$q(n) = \frac{n(n-1)}{2}$$

(21)

Proof: We will use induction for a proof. By counting systematically, comparing the last option with the other *n – 1* options takes *n - 1* comparisons. Next we compare the second to the last option with the other options except the last and this takes *n – 2* comparisons. Continue this process until we compare the second option to the first option, which takes one comparison. Consider a list of *r* options. Then

$$q(r) = (r-1)+)\ (r+2) + \cdots + 1$$

(22)

For a list of *(r +1)* options,

$$q(r+1) = r + (r-1) + \cdots + 1 = r + q(r)$$

(23)

Suppose it is true that

$$q(r) = \frac{r(r-1)}{2}$$

(24)

Then (23) becomes

$$q(r+1) = r + \frac{r(r-1)}{2} = \frac{2r + r^2 - r}{2} = \frac{(r+1)r}{2}$$

(25)

For *r* = 2, clearly there is only one comparison so the Theorem is true. If the Theorem is true for *r* = 2, it is true for *r* = 3 and by induction is it true for all *r*. This completes the proof.

## Appendix A.2

**Theorem 2**

Denote by *p(n),* the maximum number of different pairwise comparisons for a list of *n* options, containing *k* Pareto-optimal options, where *n* and *k* are integers, $2 \le k \le n$. Then *p(n)* is bounded from above by *q(n,k)*

$$p(n) \le q(n,k) = \frac{k(2n-k-1)}{2}$$
$$2 \le k \le n.$$

We prove this Theorem by induction. Consider *r+1* options in a list. The maximum number of pairwise comparisons occurs when the first *k* options in the comparisons are Pareto-optimal, and the last *r + 1 – k* options are dominated by the *k*th option. Thus

$$q(r+1,k) = [r + (r-1) \ldots + 1] - [(r-k) + \cdots + 1]$$

(26)

When there are only r options and k Pareto-optimal options,

$$q(r,k) = [(r-1) \ldots + 1] - [(r-k-1) + \cdots + 1]$$

(27)

Using Theorem 1,

$$q(r,k) = \frac{r(r-1)}{2} - \frac{(r-k)(r-k-1)}{2}$$

(28)

$$q(r+1,k) = r + \frac{r(r-1)}{2} - \frac{(r+1-k)(r-k)}{2}$$

(29)

Equation (29) can be rewritten as

$$q(r+1,k) = r + \frac{r(r-1)}{2} - \frac{(r-1-k+2)(r-k)}{2}$$
$$= r + \frac{r(r-1)}{2} - \frac{(r-1-k)(r-k)}{2} - r + k$$
$$= k + \frac{r(r-1)}{2} - \frac{(r-k)(r-k-1)}{2}$$

(30)

Using (28), (30) becomes

$$q(r+1,k) = k + q(r,k)$$

(31)

Suppose it is true that

$$q(r,k) = \frac{k(2r-k-1)}{2}$$

(32)

Then (31) becomes

$$q(k+1,k) = k + q(r,k) = k + \frac{k(2r-k-1)}{2}$$
$$q(k+1,k) = \frac{2k + k(2r-k-1)}{2} = \frac{k(2r-k+1)}{2} = \frac{k[2(r+1)-k-1}{2}$$

(33)

For *k = 2, r = 2*, there is only one pairwise comparison so, the Theorem is true. By induction the Theorem is true for *k = 2, r>2.* For *2< k = r*, all options are Pareto-optimal so that Theorem 2 is true by virtue of Theorem 1. By induction, the Theorem is true for all r >*k*. This completes the proof of Theorem 2.

## Appendix A.3

**Theorem 3**

Given any finite list of n options, each with m Performance Criteria, where m and n are integers, and where there are at least 2 Pareto-optimal options. Denote the value of the *j*th Performance Criterion for option *i* by $PC_j^i$ and denote the weighted sum $\sum_{j=1}^{m} x_j\, PC_j^i$ by $WPC^i$

$$WPC^i = \sum_{j=1}^{m} x_j\, PC_j^i$$

(34)

where $x_j \ge 0$ and $\sum_{j=1}^{m} x_j\, PC_j^i = 1$. Denote the value of *i* that results in $\max_i\, [WPC^i, i = 1,2,\ldots,n]$ by *i**. Then for each choice of the set $\{x_j\}_1^m$, *i** is a Pareto-optimal option.

Proof: Suppose the Theorem is false so that *i** is not a Pareto-optimal option. Then *i** corresponds to an inferior option because there is no dominant option. Since there is at least one Pareto-optimal option, there is an option *i*** corresponding to a Pareto-optimal option that dominates *i**, contradicting the meaning of *i**. Hence the Theorem is true.

**Appendix A.4**

**Theorem 4.**
For a specific set of weights selected from the (m – 1) – dimensional hyper triangle in Theorem 3, leading to a Pareto–optimal option *i**, there is a continuum of (m-1)-dimensional subspace of the hyper triangle, which includes the selected set of weights, such that for any set of weights in the subspace, the winning option remains as *i**.

Proof: For a specific set of weights, Theorem 3 guarantees that there is a *i** that is Pareto-optimal. To ensure that *i** is the winning option, form *(n-1)* inequalities so that *i** is better than each of the other *(n-1)* options. The intersection of the *(n-1)* half spaces with the *(m-1)*-dimensional hyper triangle yields the set of all weights that lead to *i** as the winning option. The intersection will consist of a continuum of (m – 1) dimensional subspace of the hyper triangle. Suppose that the intersection is a single point. There is no numerically feasible way to specify this weight on a real computer without any error. We exclude this single-point intersection. By construction, any selected set of weights, including the original choice of weights, in the intersection, which is a continuum of subspace in the (m-1)-dimensional hyper triangle of Theorem 3, will lead to the same Pareto-optimal option *i**. This completes the proof.